where

$$\left(\frac{2}{\bar{c}_f}\right)^{1/2}=\bar{u}_e^+,\quad \bar{y}^+=\frac{\overline{Re_\theta}}{\bar{u}_e^+}\frac{\bar{y}}{\bar{\theta}},\quad \text{and}\quad \frac{\bar{u}}{\bar{u}_e}=\frac{\bar{u}^+}{\bar{u}_e^+}$$

The parameters a and b were determined by satisfying the requirement that velocity profiles similar to those correlated by von Doenhoff and Tetervin[3] be recovered away from the wall in the outer variable $\bar{y}/\bar{\theta}$. The precise profiles established in Ref. 3 were not recovered, because of the fact that presumably more accurate data were obtained subsequent to the publication date (1943) of Ref. 3, and also because a Reynolds number effect on the velocity profiles was reported (see e.g., Ref. 6) subsequent to the appearance of Ref. 3. The correlations used were developed in Ref. 5 for two-dimensional planar flow and are steps 2 and 3 in Table 1. Table 1 is a summary of the solution to Eq. (8) for a and b at the two outer region match points of $\bar{y}/\bar{\theta}=2$ and 5.

Comparisons with Experimental Data

Experimental data are compared with Eq. (8) in Fig. 2 in the variables $\bar{u}^+$ and $\bar{y}^+$. The data of Perry[7] were taken in a decreasing adverse pressure gradient flow, and Stratford's data[7] were taken downstream of an abrupt onset of severe positive pressure gradient. Both Perry and Stratford's data are out of equilibrium and Stratford's data are near separation. Bauer's measurements[7] were made in water falling down a plate glass surface, and this boundary layer was near equilibrium. The agreement between Eq. (8) and the data in Fig. 2 is considered good. Note that there is little, if any, logarithmic region remaining in Stratford's profile.

Further comparisons with experimental data are given in Ref. 5 that include reattached boundary-layer data and variable pressure gradient compressible boundary-layer data. For application to compressible flow, the reader is referred to Ref. 5.

Summary

The analytical expression presented for the velocity distribution of a turbulent boundary layer was shown to be in good agreement with experimental data over the entire domain $0 \leq y < \infty$. The analytical result describes experimental velocity, Reynolds stress, turbulence production, and turbulence dissipation data in the region near the wall; matches correlated velocity distributions at $\bar{y}/\bar{\theta}=2$ and 5; and gives the proper limiting velocity as $\bar{y}\rightarrow\infty$. The resulting expression gives velocity explicitly as a function of $\bar{y}$ and depends on boundary-layer properties that are explicitly defined.

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Mass Matrix Correction Using an Incomplete Set of Measured Modes

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Introduction

THE comparison of dynamic analysis and dynamic test data taken on a linear lightly damped structure rarely demonstrates complete or acceptable compatibility. A number of methods have been published and discussed which assume that the analytical mass matrix is correct and then modify the measured modes to achieve orthogonality.[1-7] The assumption regarding the accuracy of the mass matrix is questionable and has been briefly discussed in a recent Technical Comment.[8] The opposite approach has also been taken which assumes that the measured modes are correct and modifies the mass matrix to achieve orthogonality.[9,10] This concept has some appeal in that the resulting analytical model (including a corrected stiffness matrix, as for example in Ref. 6) will exactly predict the results obtained in the test. Other related approaches which modify analytical matrices based on a direct comparison of predictions and measurements have been published.[11,12] These approaches are outside the scope of the present discussion, however.

As a general observation, consider three sets of data: an analytical mass matrix, an analytical stiffness matrix, and an incomplete set of measured modes. It is apparent that, if any one of these sets is assumed to be exact, it is possible to correct the other two to arrive at a model which is completely compatible with the measured data.

In Ref. 9, a rather general method of correcting the mass matrix was presented which allowed the analyst to decide which elements are to be allowed to vary and to introduce confidence factors and other external linear constraints. The method presented below is less general, especially in that all elements of the matrix will change. This method, however, is considerably simpler and will require fewer computer resources. It is probably the appropriate approach for larger problems. This method uses the method of Lagrange multipliers and the derivation was inspired by the analysis presented in Ref. 6.

Analysis

M_A is an $(n \times n)$ analytical mass matrix and Φ is an $(n \times m)$ measured modal matrix. m is the number of modes and n is the number of degrees of freedom which must correspond to the measurement points on the structure and $m<n$. The measured individual modes have been normalized so that $\Phi_i{}^T M_A \Phi_i = 1$. ΔM represents changes in the mass matrix required to satisfy the orthogonality relationship:

$$\Phi^T(M_A+\Delta M)\Phi = I$$

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or

$$\Phi^T \Delta M \Phi = I - m_A \tag{1}$$

where m_A is the nondiagonal $\Phi^T M_A \Phi$ having unit diagonal elements. Since Φ is rectangular (and has no inverse) there are an infinite number of ΔM matrices which will satisfy Eq. (1). It is possible to find that ΔM which has some minimum weighted Euclidean norm within the constraint of Eq. (1).

It is physically reasonable and mathematically convenient to minimize the function

$$\epsilon = \| N^{-1} \Delta M N^{-1} \| \tag{2}$$

where $N = M_A^{1/2}$ as in Ref. 6. Note that it is not necessary to compute N since only $N^2 = M_A$ appears in the final result.

Defining a Lagrangian multiplier λ_{ij} for each element in Eq. (1), the following Lagrangian function may be written:

$$\psi = \epsilon + \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij} (\Phi^T \Delta M \Phi - I + m_A)_{ij} \tag{3}$$

Differentiating Eq. (3) with respect to each element of ΔM and setting these results equal to zero will satisfy the minimization of Eq. (2) if the constraint of Eq. (1) is also satisfied. This process results in the matrix equation

$$2M_A^{-1} \Delta M M_A^{-1} + \Phi \Lambda^T \Phi^T = 0$$

or

$$\Delta M = -\tfrac{1}{2} M_A \Phi \Lambda^T \Phi^T M_A \tag{4}$$

where Λ is a square $(m \times m)$ matrix of λ_{ij}.
Substituting Eq. (4) into Eq. (1) allows the solution for Λ

$$\Lambda = -2 m_A^{-1} (I - m_A) m_A^{-1} \tag{5}$$

which is then substituted into Eq. (4) to obtain

$$\Delta M = M_A \Phi m_A^{-1} (I - m_A) m_A^{-1} \Phi^T M_A \tag{6}$$

Comments

Equation (6) is an easily evaluated expression for the incremental changes in the mass matrix to make it consistent with the measured modes. Note that ΔM is symmetrical as is theoretically necessary. If some minimization other than that of Eq. (2) were desired, the same process would result in a similar but probably less appealing expression than Eq. (6). If other information were available which indicated that different values of the generalized masses, $\phi_i^T (M_A + \Delta M) \Phi_i$, were more meaningful, this information would result in a diagonal matrix other than the unit matrix of Eq. (1) and the resulting ΔM would yield these values.

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Finite-Element Solution of the Supersonic Flutter of Conical Shells

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Nomenclature

$[a_I]$, $[A_I]$	= element and system aerodynamic damping matrix
$[a_R]$, $[A_R]$	= element and system aerodynamic stiffness matrix
$[k]$, $[K]$	= element and system stiffness matrix
$[m]$, $[M]$	= element and system mass matrix
M	= local Mach number
n	= number of circumferential half-waves
p	= pressure
$\{q\}$	= displacement vector $(uvw)^T$
$\{\bar{q}\}$, $\{Q\}$	= element and system nodal degrees-of-freedom vector
r	= radius
s	= meridional coordinate
S	= total shell meridional length
t	= time
u	= meridional displacement
v	= tangential displacement
V	= local air velocity
w	= radial displacement
β	= rotational nodal degree of freedom
δ	= variational operator
θ	= circumferential coordinate
κ	= reduced frequency, $\omega S/V$
λ	= dynamic pressure parameter, $\rho_a V^2 R_1^3 / D(M^2-1)^{1/2}$
ρ_a	= air mass density
ω	= frequency, $\omega_R + i\omega_I$, $i = (-1)^{1/2}$

Subscripts

a	= air
cr	= critical

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